cyclic shifts of these. Thus p must be λ . Similarly $\text{sig}(X, \lambda X) = (0, \bar{\omega}, \omega, \omega, 1, \bar{\omega}, 1)$ must equal $\text{sig}(\Lambda X, qY)$. Hence p equals $\omega\lambda$.

Theorem 5: There is no permutation of order 7 in the group of a (24, 12, 10) quaternary code.

Proof: By the previous discussion such a code is equivalent to one of the following form:

$$\begin{array}{ccccccc} (0_7 & X & \Lambda X)_3 & 0 & 0 & 0 & \\ (\Lambda X & X & 0_7)_3 & 0 & 0 & 0 & \\ (Y & \lambda Y & \omega\lambda Y)_3 & 0 & 0 & 0 & \\ h_7 & 0_7 & 0_7 & 1 & \omega & \omega & \\ 0_7 & h_7 & 0_7 & \omega & 1 & \omega & \\ 0_7 & 0_7 & h_7 & \omega & \omega & 1 & \end{array}$$

The block of nine zeros on the upper right refer to the final three coordinates (which are zero) of the nine vectors to their left.

The CAMAC computer system quickly found vectors of weight 8 in the code. Q.E.D.

The code above looks interesting, and so we found its group and weight distribution by computer. The weight distribution follows:

$$\begin{aligned} A_8 &= 3(252) \\ A_{10} &= 3(4,056) \\ A_{12} &= 3(59,220) \\ A_{14} &= 3(368,424) \\ A_{16} &= 3(1,263,159) \\ A_{18} &= 3(2,068,584) \\ A_{20} &= 3(1,473,444) \\ A_{22} &= 3(344,088) \\ A_{24} &= 3(11,178). \end{aligned}$$

This code cannot be the quaternary code generated by the vectors in the binary Golay (24, 12, 8) code as this latter code would have at least 3(759) vectors of weight 8. The group of this code has order 63. On the coordinates there are three seven-cycles and one three-cycle and the group is not transitive.

Corollary: A monomial M of order 7 cannot be in the group of a (24, 12, 10) quaternary self-dual code.

Proof: The proof follows as for the corollary to Theorem 4 using Theorem 5 in place of Theorem 4.

VI. ANOTHER PROOF FOR A PRIME 3 CASE

In [2] there is only one situation open for a permutation of order 3, one with eight cycles and no fixed points. Preliminary investigations indicate that there are more codes to examine than for the 11 and 7 permutations considered in the previous sections and this problem is still open. We give here a simpler proof than in [2] for the case where there is a permutation of order 3 with seven cycles and three fixed points.

Then $\bar{D}$ is a self-dual (10, 5) code by [2, th. 3]. Further, $\bar{D}$ must be indecomposable or it would have a summand of weight 2 yielding a weight 6 vector in C . Hence by [5], $\bar{D}$ must be either E_{10} or B_{10} .

A generator matrix for B_{10} is as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \omega & \bar{\omega} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \bar{\omega} & 1 \\ 0 & 1 & \bar{\omega} & \omega & 0 & 0 & 1 & \bar{\omega} & \omega & 0 \end{pmatrix}.$$

Two of the fixed points must be on one side; thus there is a

vector in $\bar{D}$ of weight 4 with nonzero coordinates on them yielding a vector of weight 8 or less in C , a contradiction.

We give only the first four rows of a generator matrix of E_{10} as we need not consider the last row:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The ten coordinates are in five pairs such that the union of two pairs holds a weight 4 vector in E_{10} with all one components. Of the three fixed points at least two are in two pairs, the union of which holds a weight 4 vector in $\bar{D}$. This gives rise to a vector of weight 8 or less in C and is thus impossible.

VII. ACKNOWLEDGMENT

We want to thank Thom Grace for his ingenious programming help. We also thank Jefferey Leon for the use of his code automorphism group program. We thank Norman Hamilton for stimulating discussions and for pointing out Lemma 1. The CAMAC computer system at the University of Illinois at Chicago Circle, Computing Center was used extensively in the computations performed in this work.

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Compressors for Combined Source and Channel Coding with Applications to the Generalized Gaussian Family

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Abstract—The twin tools of asymptotic quantization theory and rate-distortion theory are used to find an expression for the approximate total

Manuscript received December 17, 1980; revised April 15, 1982. This research was supported in part by the Air Force Office of Scientific Research under Grants AFOSR-76-3062 and AFOSR-81-0047; in part by Consejo Nacional de Ciencia y Tecnología, Mexico; and in part by the Department of Defense Joint Services Electronics Program under Contract F49620-77-C-0101.

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mean-squared distortion of the series connection of a nonuniform quantizer and a noisy channel. The nonuniform quantizer is modeled as a compressor followed by a uniform quantizer. The approximate total mean-squared error distortion is minimized over a class of parameterized compressor characteristics for input processes whose univariate probability density functions are members of the generalized Gaussian family.

I. INTRODUCTION

The quantization of continuous amplitude, discrete-time signals combined with the transmission of the quantized samples over noisy channels is a problem that has been studied by several authors. From an information theoretic point of view, Shannon [1], [2] set bounds on the performance achievable by block codes, and Gray and Ornstein [3] studied bounds on the performance if nonblock structures are used. Berger [4] explicitly considered the joint source-channel coding problem making use of the results of [2]. This problem has also been studied from an implementation point of view. Kurtenbach and Wintz [5] used a mean-squared error criterion to optimally design the quantizer structure based on the input probability density function and the transition probabilities of a discrete memoryless channel. Ancheta [6] studied codes that are suitable for error correction as well as for data compression. Buchner [7] analyzed the effect of channel errors on Gray-coded pulse code modulation (PCM), while Essman and Wintz [8] studied differential pulse code modulation (DPCM) contaminated with channel noise. They also compared PCM with DPCM performance if they are both transmitted over a noisy channel. Applications of combined source-channel coding schemes have been analyzed by Modestino and Daut [9], who worked on the problem in an image-processing environment.

In this correspondence we propose to investigate the total mean-squared distortion suffered by a compressed, continuous amplitude memoryless source that is uniformly quantized and transmitted over a noisy channel with a known capacity. We are interested in a small distortion analysis—i.e., quantizers with very large numbers of quantization levels and channels whose capacities are large enough to carry the data rates coming out of the quantizer. To study the quantizer distortion we use Bennett's model for one-dimensional quantizers [10]: a zero-memory nonlinearity $G(x)$ taking values in $[0, 1]$, called the compressor, which is invertible and continuously differentiable on the support of the input distribution $F_X(x)$, followed by a uniform N -level quantizer on $[0, 1]$. The quantizer's output is in turn connected to the inverse nonlinearity $G^{-1}(y)$. We will let $g(x) = dG(x)/dx$, and we will assume that the support of $F_X(x)$ is a subset of the support of $g(x)$. Furthermore, we will assume that $F_X(x)$ is absolutely continuous, and we will denote $dF_X(x)/dx$ by $f_X(x)$. Finally, we will also assume that

$$\lim_{N \rightarrow \infty} N^2 \text{mse}_N = \frac{1}{12} \int [g(x)]^{-2} dF_X(x) < \infty,$$

where mse_N is the mean-squared error for an N -level quantizer. All the integrals will be taken over the interval $(-\infty, \infty)$. We note in passing that sufficient conditions for the last assumption are presented in [11]. Since we are interested in small quantization distortion, we have a large number N of quantization levels and we will approximate the mean-squared error by [10]

$$\epsilon_q^2 \approx \frac{1}{12N^2} \int [g(x)]^{-2} f_X(x) dx. \quad (1)$$

For the analysis of the channel distortion we use some results from rate-distortion theory. The difference between the Shannon lower bound $R_L(D)$ and the rate distortion function $R(D)$ converges to zero as $D \rightarrow 0$ for a broad class of source distributions and difference distortion measures [12, ch. 4]. Since we want to analyze small distortions, we approximate the rate-distortion function for the mean-squared error fidelity criterion by the Shannon lower bound. If we are given a source with differential entropy H_X and a channel with capacity C , then the Shannon lower bound tells us that with an optimal code, a good approximation to the mean-squared distortion between channel input

Fig. 1. Combination of a nonuniform quantizer and a noisy channel.

and output is given by [12, p. 98]

$$\sigma_{ch}^2 \approx \frac{e^{-2(C-H_Y)}}{2\pi e}. \quad (2)$$

We remark that H_Y can sometimes be difficult to calculate in practice. In the case of a Gaussian source with variance σ^2 , the Shannon lower bound and the rate-distortion function coincide over the range $0 \leq D \leq \sigma^2$ if mean-squared error is the fidelity criterion. If an adequate channel coder is used and the input data are grouped into sufficiently large blocks, then the mean-squared distortion of the source S , which is compressed before the channel and expanded after the channel, can be expressed as

$$\epsilon_{ch}^2 \approx \sigma_{ch}^2 \int \frac{f_X(x)}{[g(x)]^2} dx.$$

We assume that this last expression holds, and we will develop our model in the following sections.

II. DEVELOPMENT

Consider a nonuniform quantizer consisting of the series connection of the compressor $G(x)$ and a uniform N -level quantizer. The quantizer mean-squared error is measured between the input to the compressor and the output of the expander $G^{-1}(y)$. For the uniform quantizer, we will assume that the noise resulting from the uniform quantization has zero mean; for large N the noise is approximately uniform in distribution over the length of the quantization interval $1/N$. So we will interpret the error given by (1) as the expanded variance of the error of a uniform N -level quantizer. The expansion takes place in the inverse function $G^{-1}(y)$.

If we transmit the output of the quantizer through a noisy channel as shown in Fig. 1, then the quantization noise as well as the channel noise additively contaminate the information bearing signal between the compressor and the expander. We will assume that these noise processes are uncorrelated and that the channel noise has zero mean. Then there exists a code such that the total mean-squared error between the outputs of the compressor and the channel sequence $\hat{y}$ is

$$\sigma_T^2 = \sigma_{ch}^2 + \sigma_q^2, \quad (3)$$

where σ_q^2 is given by $1/(12N^2)$ and σ_{ch}^2 , the input dependent channel distortion, is given by (2). The effect of the system depicted in Fig. 1 on this noise process can now be computed from Bennett's integral (1). This yields the total mean-squared distortion between the source output and the expander output, which is

$$d = \int \frac{f_X(x)}{[g(x)]^2} dx \left[\frac{1}{12N^2} + \frac{e^{-2(C-H_Y)}}{2\pi e} \right]. \quad (4)$$

Now we ask how large must N be so that the quantizer has little influence on the channel distortion. This distortion is a function only of the differential entropy of the channel input. If H_X is the differential entropy of the source and C is the channel capacity, then the inequality $\log N \gg C \gg H_X$ guarantees this condition. These conditions, while stringent, can be satisfied by many useful communication systems. So we approximate the channel input by a memoryless source with a differential entropy given by

$$H_Y = H_X + E[\log g(X)], \quad (5)$$

where the second summand is due to the compression of the

original source. If we substitute (5) into (4) we obtain

$$d = \int f_X(x) [g(x)]^{-2} dx \left[\frac{1}{12N^2} + \frac{1}{2\pi e} \exp \left(-2C + 2H_X + 2 \int f_X(x) \log g(x) dx \right) \right]. \quad (6)$$

Defining

$$K = \frac{1}{12N^2} \quad (7)$$

and

$$L = \frac{e^{-2C+2H_X}}{2\pi e}, \quad (8)$$

we find that the total mean-squared error in the system is given by

$$d = \int f_X(x) [g(x)]^{-2} dx \left[K + L \exp \left(2 \int f_X(x) \log g(x) dx \right) \right]. \quad (9)$$

We observe that if $L = 0$, an infinite capacity channel, d as given in (9) reduces to Bennett's integral (1). It is known that the compressor characteristic which minimizes the distortion in the absence of channel noise is

$$g(x) = \frac{[f_X(x)]^{1/3}}{\int [f_X(u)]^{1/3} du}. \quad (10)$$

Since in our problem the channel is not ideal, we suggest the use of *ad hoc* compressors given by

$$g(x) = \frac{[f_X(x)]^a}{\int [f_X(u)]^a du} \quad (11)$$

where $a \in (0, 1/2)$. We note that under mild conditions $[f_X(x)]^a$ is integrable (i.e., $G(x)$ is well-defined); for example, this is implied by the existence of all the moments of X .

A class of probability densities often used in communications is the zero-mean, unit variance generalized Gaussian family given by

$$f_X(x) = \gamma(p) \exp \left\{ -\frac{|x|^p}{A(p)} \right\}, \quad p \geq 0, \quad (12)$$

where $A(p) = [\Gamma(1/p)/\Gamma(3/p)]^{1/2}$,

$$\gamma(p) = \frac{p}{2A(p)\Gamma(1/p)},$$

and where $\Gamma(\cdot)$ is the gamma function. The parameter p regulates the rate of decay of the exponential and this family can be used to study the influence of the shape of the density on the performance of the system [13], [14]. For the compressor family given by (11), the generalized Gaussian family satisfies all of our standing assumptions on the input [11]. The special cases $p = 1$ and $p = 2$ yield Laplace and Gaussian densities, respectively.

Our goal is now to minimize the total system distortion (9) using the parameterized class of compressors (11), when the input density $f_X(x)$ is a member of the family (12). The design variable is the parameter a appearing in (11). A necessary condition for a to minimize d is that $\partial d / \partial a = 0$ at the optimizing a . Sufficiency is obtained if $\partial^2 d / \partial a^2 > 0$ at the optimum a . We analyze these conditions in the next section.

III. RESULTS

Theorem: Consider the family of generalized Gaussian densities (12) and the class of parameterized compressors (11). The value of the parameter a that minimizes the distortion (9) is contained in the interval $(0, 1/3)$.

Proof: A necessary condition for the parameter a to minimize d is that $\partial d / \partial a = 0$ at the optimum a . If we substitute the expression for $g(x)$ into the expression for d we obtain

$$d = \int [f_X(x)]^{1-2a} dx \left[K \left(\int [f_X(x)]^a dx \right)^2 + L \exp \left\{ 2a \int f_X(x) \log f_X(x) dx \right\} \right].$$

Now we use the family of generalized Gaussian densities and the integrals [15]

$$\int [f_X(x)]^B \log f_X(x) dx = \gamma(p)^{B-1} B^{-1/p} \left[\log \gamma(p) - \frac{1}{Bp} \right]$$

$$\int [f_X(x)]^B dx = \gamma(p)^{B-1} B^{-1/p}$$

to see that

$$\frac{\partial d}{\partial a} = \frac{a}{p(1-2a)^{1+1/p}} \cdot \left[-2Le^{-2a/p} + K[\gamma(p)]^{-2} a^{-2(1+1/p)} (1-3a) \right]. \quad (13)$$

Setting $\partial d / \partial a = 0$ yields

$$\frac{K}{L} = \frac{2\gamma^2(p)e^{-2a/p}a^{2(p+1)/p}}{1-3a}. \quad (14)$$

Since K , L and the numerator of (14) are positive, the denominator of (14) is also positive, and we conclude that the value of a that minimizes d lies within the interval $(0, 1/3)$.

We now look at the sufficiency by determining the sign of $\partial^2 d / \partial a^2$. Taking the derivative of (13) with respect to a , we get

$$\frac{\partial^2 d}{\partial a^2} = (1-2a)^{-2-1/p} \left\{ \frac{4aLe^{-2a/p}}{p} \left(\frac{4a}{p} + \frac{1}{a} \right) + K \frac{a^{-2(1+1/p)}}{\gamma^2(p) \cdot p} \left[\frac{4}{p} (6a^2 - 5a + 1) + 3a^2 - 5a + 2 \right] \right\}.$$

It can easily be verified that $a \in (0, 1/3)$ implies that $\partial^2 d / \partial a^2 > 0$. Thus the value of $a \in (0, 1/3)$ satisfying (14) minimizes the distortion d . Q.E.D.

Corollary: For a given p and ratio K/L , the value of the parameter a that minimizes d is unique.

Proof: We take the derivative of K/L as given in (14), with respect to a , i.e.,

$$\frac{\partial(K/L)}{\partial a} = \frac{2\gamma^2(p)a^{1+2/p}}{p(1-3a)e^{2a/p}} [6a^2 - a(3p+8) + 2(p+1)].$$

Since the term inside the brackets has no roots in the interval $(0, 1/3)$, it follows that K/L is a monotonic function of a in this interval, from which we conclude that the optimum value of a is unique. Q.E.D.

It is interesting to notice that for a given density the value of a that minimizes d is only a function of the ratio between quantization noise and channel noise. In Fig. 2 we present plots illustrating the dependency between the parameter a and the ratio K/L .

Fig. 2. Parameter a versus ratio K/L for the indicated distributions.

for several densities of the generalized Gaussian family. We note that for small K/L , the optimizing value of the compressor exponent a is very sensitive to the input density parameter p . The total mean-squared distortion obtained by using the optimal compressor for $p = 1.5$, with the number of quantizer output levels N as a parameter, is plotted as a function of the ratio K/L in Fig. 3. In [16] appear analogous plots for $p = 1$ and $p = 2$.

If we now go back to (2) and (5), use the definition of L , and the family of compressors (11), we see that the channel distortion is given by

$$\sigma_{ch}^2 = L\gamma(p)^2 a^{2/p} d^{-2a/p}. \quad (15)$$

Assume now that the magnitudes of the noises are related by $\sigma_{ch}^2 = \alpha \sigma_q^2$, where α is a positive constant and $\sigma_q^2 = K = (12N^2)^{-1}$. Then using (15) we see that

$$\frac{K}{L} = \frac{1}{\alpha} \gamma^2(p) a^{2/p} e^{-2a/p}. \quad (16)$$

However, for the optimal compressor, K/L is given by (14). Combining (14) and (16) we see that

$$\alpha = \frac{1 - 3a}{2a^2}$$

In the case of equal noise powers, i.e., $\alpha = 1$, this last equation yields $a \approx 0.2808$. Comparing this value with $1/3$, which is obtained if the channel noise is not considered in the compressor design, we observe that the channel has little influence in the compressor characteristic if the noise powers are equal. Larger values of α , however, yield compressors which are significantly different from the $a = 1/3$ case. For example, $\alpha = 10$ yields $a = 0.1608$ and with $\alpha = 20$, a value of $a = 0.1250$ is obtained.

IV. EXAMPLES

In this section we compare the performance of the optimal compressor in the case where the channel noise is not considered in the design (i.e., if $a = 1/3$) and in the case where the signal is not compressed. Let d_{nc} , $d_{\min}$, and $d_{1/3}$ be the distortions obtained with no compression, with the optimal compressor and with $a = 1/3$, respectively.

Example 1: Let X be the output of a Laplace source (i.e., let $p = 1$ in (12)), and let $N = 512$.

- a) Consider the case where $\alpha = 1$. From our expression for d we find that

$$d_{\min} = 3.68 \times 10^{-5}.$$

Fig. 3. Approximate total mean-squared distortion for various numbers of quantization levels and $p = 1.5$.

On the other hand,

$$d_{1/3} = 3.91 \times 10^{-5}.$$

For the no-compression case, we neglect the probability mass outside the interval $[-6, 6]$ and obtain

$$d_{nc} = 1.11 \times 10^{-4}.$$

We can see that $d_{\min}$ represents improvements of approximately 5.6 percent and 66.8 percent over $d_{1/3}$ and d_{nc} , respectively.

- b) Consider now the case $\alpha = 10$. Here,

$$d_{\min} = 3.98 \times 10^{-4}$$

$$d_{1/3} = 5.39 \times 10^{-4}$$

$$d_{nc} = 6.09 \times 10^{-4}$$

and the improvements are approximately 26.2 percent and 34.7 percent over $d_{1/3}$ and d_{nc} , respectively.

Example 2: Let X be the output of a Gaussian source ($p = 2$ in (12)). Let again $N = 512$.

- a) Let $\alpha = 1$. Then

$$d_{\min} = 2.15 \times 10^{-5}$$

$$d_{1/3} = 2.21 \times 10^{-5}$$

$$d_{nc} = 4.07 \times 10^{-5}.$$

(We neglect the probability mass outside $[-4, 4]$ for the no compression case.) The performance improvements are approximately 2.7 percent and 47.2 percent over $d_{1/3}$ and d_{nc} , respectively.

- b) Let $\alpha = 10$. This yields

$$d_{\min} = 1.66 \times 10^{-4}$$

$$d_{1/3} = 1.91 \times 10^{-4}$$

$$d_{nc} = 2.24 \times 10^{-4}.$$

The improvements are 13.1 percent and 25.8 percent.

From these examples we can see that for equal noise powers ($\alpha = 1$), the optimum value of the parameter a yields a large improvement over the no-compression quantizer, while the improvement over the $a = 1/3$ compressor is relatively small. On the other hand, with $\alpha = 10$, the optimal compressor has significant performance improvements for both cases.

V. CONCLUSION

For small quantizer distortion-channel distortion ratios the value of a in our parameterized class of compressor characteristics is very small, which corresponds to very little compression. Berger [12, ch. 5] points out that the best quantizer for a source that is going to be entropy coded is asymptotically the uniform quantizer (no compression). Since by using Shannon's lower bound to the rate distortion function we are essentially considering only signals which have been optimally entropy coded, our results coincide with this observation. We can also verify in the examples that even though our parameterized optimal compressor always reduces the distortions achievable with no-compression and with the $a = 1/3$ compressor, the improvement depends on the particular distribution of the source output and on the relative magnitudes of the quantizer noise and the channel noise.

ACKNOWLEDGMENT

We would like to thank Nahid Khazenie, who made the plots in this correspondence.

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A Fast Algorithm for Determining the Complexity of a Binary Sequence with Period 2^n

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Abstract—The complexity of a periodic sequence (s) is defined to be the least number of stages in a linear feedback shift register that generates (s).

Manuscript received September 8, 1980; revised February 12, 1982. This work was presented at the Sixteenth OSU-Denison Mathematics Conference, Denison University, Granville, OH, May 7, 1981.

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Fig. 1. An N -stage LFSR.

A fast algorithm for determining the complexity of a binary sequence with period 2^n is described. The algorithm uses the properties of the basis sequences $\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ j \end{pmatrix}, \begin{pmatrix} 1 \\ j \end{pmatrix}, \begin{pmatrix} 2 \\ j \end{pmatrix}, \dots$.

I. INTRODUCTION

Let $s = (s_0, s_1, \dots, s_{N-1})$ be a binary vector of length N , and let the sequence (s) be defined by appending copies of s ; that is, for $i \geq 0$, $(s)_i = s_{i \pmod{N}}$. Then (s) is a periodic sequence with period N (although if s itself is periodic, then the sequence (s) may have a smaller period). The sequence (s) could be generated by the N -stage cyclic linear feedback shift register (LFSR), which is diagrammed in Fig. 1, using the initial loading $s = (s_0, s_1, \dots, s_{N-1})$. The complexity of (s), denoted by $c(s)$, is defined as the least number of stages in a linear feedback shift register (LFSR) that generates (s) [1], [2]. In this case, $c(s) \leq N$. In general, $c(s)$ can be determined using an algorithm developed by Massey [3]. In the case that N is a power of 2, the complexity can be determined in $\log N$ steps using a much simpler algorithm presented in this correspondence. The algorithm was developed to study the complexities of sequences of maximum period 2^n generated by an n -stage nonlinear feedback shift register [4, pp. 128-141].

In the next section, the algorithm is presented, and in Section III it is justified. In the concluding remarks, the two algorithms are compared.

II. THE ALGORITHM FOR COMPUTING THE COMPLEXITY OF A BINARY SEQUENCE OF PERIOD 2^n

Let $a = (a_0, a_1, \dots, a_{N-1})$ be a vector of length $N = 2^n$, $n > 0$. Decompose a into its left and right halves by

$$L(a) = (a_0, a_1, \dots, a_{N/2-1})$$

$$R(a) = (a_{N/2}, a_{N/2+1}, \dots, a_{N-1}),$$

and write $a = (L(a) : R(a))$.

Suppose (s) = $(s_0, s_1, \dots, s_i, \dots)$ is a periodic binary sequence with period $N = 2^n$, $n > 0$, and let $s = (s_0, s_1, \dots, s_{N-1})$ denote a single period of (s). The algorithm, which is represented in Fig. 2 by its flowchart, accepts the vector s and produces the complexity $c(s)$, which is stored in the variable c , after $n = \log N$ iterations.

Suppose, for example,

$$s = 1101100011010011011110011111010$$

is given with length 32. The algorithm performs the following steps:

Initializations: $a \leftarrow s$; $l \leftarrow 32$; $c \leftarrow 0$

		l	c
Step 1)	$R(a) = 0111100111111010$ $L(a) = 1101100011010011$ $b = 1010000100101001$ $a \leftarrow b$	16	16
Step 2)	$R(a) = 00101001$ $L(a) = 10100001$ $b = 10001000$ $a \leftarrow b$	8	24
Step 3)	$R(a) = 1000$ $L(a) = 1000$ $b = 0000$ $a \leftarrow L(a)$	4	24
Step 4)	$R(a) = 00$ $L(a) = 10$ $b = 10$ $a \leftarrow b$	2	26