Now taking the vec operation on both sides of (4.11) and (4.20), we have

$$\text{vec } \dot{D}_{ji}(t) = (I \otimes A_j) \text{vec } D_{ji}(t) + (I \otimes B_j) \text{vec } F_i(t) \\ = (I \otimes A_j) \text{vec } D_{ji}(t) + (I \otimes B_j) \mathbb{E} \mathbb{B}^T \mathcal{P}_i(t) \quad (\text{A.2.2})$$

$$\text{vec } \dot{\Gamma}_{ji}(t) = -(I \otimes A_j^T) \text{vec } \Gamma_{ji}(t) + 2[(C_j K_{ji} C_j^T + R_{vj}) \otimes Q_j] \\ \cdot \text{vec } D_{ji}(t) + 2 \text{vec } [Q_j \exp(A_j t) K_{ji} C_j]. \quad (\text{A.2.3})$$

Introducing (A.2.2), (A.2.3) in $\ddot{\mathcal{X}}_i$ and $\dot{\mathcal{P}}_i$ with $\mathcal{X}_i$ and $\mathcal{P}_i$ as defined in (5.6), (5.7), we have

$$\ddot{\mathcal{X}}_i(t) = \mathbb{A} \mathcal{X}_i(t) + \mathbb{B} \mathbb{E} \mathbb{B}^T \mathcal{P}_i(t) \quad (\text{A.2.4})$$

$$\dot{\mathcal{P}}_i(t) = \mathbb{G} \mathcal{X}_i(t) - \mathbb{A}^T \mathcal{P}_i(t) + \mathbb{U}_i(t). \quad (\text{A.2.5})$$

Then (5.8) can be easily established from (A.2.4), (A.2.5). Thus, we have shown the TPBVP (4.11), (4.20), (4.22) implies (5.8). Conversely, we can show that (4.22), (5.8) imply (4.11) and (4.20).

Next, we show that the TPBVP (5.8), (5.9) has a unique solution. It has been shown in [9] that TPBVP's of the form (5.8) have a unique solution for almost all $T_1, T_2, \dots, T_d$ if and only if the pair $(\mathbb{A}, \mathbb{B} \mathbb{E} \mathbb{B}^T)$ is controllable. Note that $\mathbb{E} > 0$, because by assumption $R_{vj} > 0$. Then it can be shown easily that $(\mathbb{A}, \mathbb{B} \mathbb{E} \mathbb{B}^T)$ is controllable iff $(\mathbb{A}, \mathbb{B})$ is controllable. Due to Lemma A.2, the latter is true if and only if $(\mathcal{A}, \mathcal{B})$ is controllable. Thus, (5.8) has a unique solution under the previous assumptions.

Now it remains to show the solution to (5.8) is indeed the unique optimizer. We rewrite the cost as

$$J = \sum_{j=1}^N \sum_{i=1}^d \left\{ \int_0^{T_i} \text{tr} \left[K_{ji} (\exp(A_j t) + D_{ji} C_j)^T \right. \right. \\ \cdot Q_j (\exp(A_j t) + D_{ji} C_j) + R_{vj} D_{ji}^T Q_j D_{ji} \Big] dt \\ + \int_0^{T_i} \text{tr} \left[K_{ji} C_j^T F_i^T R_j F_i C_j + R_{vj} F_i^T R_j F_i \right] dt \\ \left. + \int_0^{T_i} \text{tr} [Q_j R_{\omega_j}] dt \right\}. \quad (\text{A.2.6})$$

We can easily show that the first integrand is convex in $D_{ji}(\cdot)$ and the second convex in $F_i(\cdot)$. Now the problem is to minimize the cost in (A.2.6) under the constraint (4.11) with the fixed initial and final values. It follows from the results of [14] that the hold functions satisfying (4.11), (4.20), (4.22), and the chosen final value $D_{ji} T_i$ indeed achieve the minimal cost of the fixed monodromy problem. ■

REFERENCES

- [1] A. E. Bryson and Y.-C. Ho, *Applied Optimal Control*. New York: Hemisphere, 1975.
- [2] A. B. Chammas and C. T. Leondes, "On the design of linear time-invariant systems by periodic output feedback, Part I and II," *Int. J. Contr.*, vol. 27, no. 6, pp. 885-903, 1978.
- [3] —, "Optimal control of stochastic linear systems by discrete output feedback," *IEEE Trans. Automat. Contr.*, vol. AC-23, no. 5, pp. 921-926, 1978.
- [4] —, "On the finite time control of linear systems by piecewise constant output feedback," *Int. J. Contr.*, vol. 30, no. 2, pp. 227-234, 1979.
- [5] G. F. Franklin and J. D. Powell, *Digital Control of Dynamic Systems*. Reading, MA: Addison-Wesley, 1980.
- [6] B. K. Ghosh and C. I. Byrnes, "Simultaneous stabilization and simultaneous pole-assignment by nonswitching dynamic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 735-741, 1983.
- [7] Y.-C. Juan and P. T. Kabamba, "Constrained structure control of periodic systems using generalized sampled-data hold functions," in *Proc. 1987 ACC*, Minneapolis, MN, 1987, pp. 1695-1700.

- [8] —, "Optimal hold functions for sampled-data regulation," in *Proc. 1988 ACC*, Atlanta, GA, 1988, pp. 884-885.
- [9] Y.-C. Juan, "Control of linear systems using generalized sampled-data hold function control," Ph.D. dissertation, Univ. Michigan, Ann Arbor, MI, 1988.
- [10] P. T. Kabamba, "Monodromy eigenvalue assignment in linear periodic systems," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 10, pp. 950-952, 1986.
- [11] —, "Control of linear systems using generalized sampled-data hold functions," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 9, pp. 772-783, 1987.
- [12] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1088-1096, 1985.
- [13] P. P. Khargonekar, A. Pascoal, and R. Ravi, "Stabilization of linear time-invariant plants: Strong, simultaneous, and reliable stabilization," in *Proc. 1988 ACC*, Atlanta, GA, 1988, pp. 2477-2482.
- [14] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*. New York: Wiley, 1967.
- [15] K. D. Minto and M. Vidyasagar, "Extension of simultaneous stabilization to more than 2 plants," *Contr. Theory Advanced Technol.*, vol. 2, no. 1, pp. 39-64, 1986.
- [16] A. W. Olbrot, "Robust stabilization of uncertain systems by periodic feedback," *Int. J. Contr.*, vol. 45, no. 3, pp. 747-758, 1987.
- [17] R. Sacks and J. Murray, "Fractional representation, algebraic geometry, and the simultaneous stabilization problem," *IEEE Trans. Automat. Contr.*, vol. AC-27, no. 4, pp. 895-903, 1982.
- [18] T. J. Tarn, J. R. Zavgren, and X. Zeng, "Stabilization of infinite-dimensional systems with periodic feedback gains and sampled output," *Automatica*, vol. 24, no. 1, pp. 95-99, 1988.
- [19] —, "Feedback stabilization of linear dynamic systems with multi-rate sampled output," *SIAM J. Contr. Optimiz.*, vol. 26, no. 4, pp. 812-833, 1988.
- [20] T. J. Tarn and T. Yang, "Simultaneous stabilization of infinite-dimensional systems with periodic output feedback," in *Linear Circuits, Systems and Signal Processing: Theory and Application*. New York: Elsevier, 1988, pp. 409-424.
- [21] M. Vidyasagar and N. Viswanadhan, "Algebraic design techniques for reliable stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, no. 5, pp. 1085-1095, 1982.
- [22] M. Vidyasagar, *Control Systems Synthesis: A Factorization Approach*. Cambridge, MA: M.I.T. Press, 1985.
- [23] C. Yang and P. T. Kabamba, "Simultaneous design of linear time-invariant systems using generalized sampled-data hold functions," in *Proc. 1988 ACC*, Atlanta, GA, 1988, pp. 784-789.
- [24] C. Yang, "Control of linear systems with plant uncertainties using generalized sampled-data control techniques," Ph.D. dissertation, Univ. Michigan, Ann Arbor, MI, to be published.

On the Spectral Set for a Family of Polynomials

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Abstract—Given a family of real n th-order polynomials P , the spectral set of P is defined as

$$\sigma(\mathcal{P}) \doteq \{z \in \mathbb{C} : p(z) = 0 \quad \text{for some } p(\cdot) \in \mathcal{P}\}.$$

The focal point of this note is the problem of generating the spectral set for two different families of polynomials P —spherical families and polytopic families. For the spherical case, a closed form equation describing the boundary of the spectral set is obtained. In the polytopic

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case, we show a computationally feasible technique which requires only a two-dimensional gridding of a bounded subset of the complex plane rather than a high-dimensional gridding of the parameter set. Two numerical examples conclude the note.

I. INTRODUCTION

Generally speaking, much of the recent robust stability literature is concerned with the following problem: given a family of polynomials $\mathcal{P}$ and a region $\mathcal{D}$ in the complex plane, determine if all polynomials in $\mathcal{P}$ have all their roots in $\mathcal{D}$ (e.g., see [1] for a review of this literature). The motivation for this note is a desire to know something more about the "root distribution" of $\mathcal{P}$ rather than a simple "yes" or "no" answer whether all roots are in $\mathcal{D}$. From a control systems point of view, knowing the root distribution makes it possible to analyze robust performance—not only robust stability. This leads to the problem of the spectral set generation.

More formally, when all polynomials in $\mathcal{P}$ have the same degree, the *spectral set* of $\mathcal{P}$ is defined as

$$\sigma(\mathcal{P}) \doteq \{z \in \mathbb{C}: p(z) = 0 \text{ for some } p(\cdot) \in \mathcal{P}\}. \quad (1)$$

Note that once $\sigma(\mathcal{P})$ has been found, the robust stability problem is solved for all possible $\mathcal{D}$ regions. That is, given an arbitrary $\mathcal{D}$ region, robust stability is guaranteed if and only if

$$\sigma(\mathcal{P}) \subseteq \mathcal{D}. \quad (2)$$

Hence, the problem of spectral set generation can be viewed as a generalization of the $\mathcal{D}$ -stability problem.

The main objective of the note is to describe viable computational techniques to generate $\sigma(\mathcal{P})$ for two types of polynomial families $\mathcal{P}$ —spherical families and polytopic families. By "viable," we mean that gridding the so-called *perturbation set* is avoided. To elaborate on this point, suppose that the family $\mathcal{P}$ is obtained as some *perturbation vector* q ranges over a given *bounding set* $Q \subseteq \mathbb{R}^l$. That is, associated with each $q \in Q$ is a polynomial $p(s, q)$. Therefore, we write

$$\mathcal{P} = \{p(s, q): q \in Q\}. \quad (3)$$

Now, the "gridding approach" leads to approximation of $\sigma(\mathcal{P})$ as follows: first, generate some suitably fine gridding $\{q^i\}$ of the bounding set Q . Then, for each gridpoint q^i , compute the roots of the polynomial $p(s, q^i)$. The approximation of the spectral set is then the collection of all such roots, i.e., if $\sigma(p(s, q^i))$ denotes the root set for $p(s, q^i)$, then

$$\sigma(\mathcal{P}) \approx \bigcup_i \sigma(p(s, q^i)). \quad (4)$$

Notice, however, that this naive approach leads to a combinatoric explosion in computation, e.g., if Q is a rectangle in $\mathbb{R}^n$ and each edge of Q has N gridpoints, then the roots of N^n polynomials must be found. This combinatoric explosion is the issue which the techniques in the note circumvent for both spherical and polytopic families; these families are now defined.

II. SPECTRAL SET PROBLEM FOR SPHERICAL AND POLYTOPIC FAMILIES

The Spherical Problem: Given a radius $r \geq 0$, a *spherical family* of polynomials $\mathcal{P}$ is described by

$$p(s, q) \doteq p_0(s) + \sum_{i=0}^n q_i s^i \quad (5)$$

where $p_0(s)$ is a fixed n th-order polynomial and q ranges in the sphere

$$q \in Q_r \doteq \{q: q^T q \leq r^2\}. \quad (6)$$

Hence, we obtain the family of polynomials

$$\mathcal{P} = \{p(s, q): q \in Q\}. \quad (7)$$

Throughout this note, it is assumed that all polynomials in $\mathcal{P}$ have degree equal to n . Hence, if a_n denotes the coefficient of s^n in $p_0(s)$, this assumption is equivalent to

$$|a_n| > r. \quad (8)$$

Note that the aforementioned degree requirement is automatically satisfied whenever $p(s, q)$ is monic.

Remarks: It will be shown that the spectral set $\sigma(\mathcal{P})$ admits a closed form characterization. That is, we will construct a function $\Phi(z)$ having the following property: $z \in \sigma(\mathcal{P})$ if and only if

$$\Phi(z) \leq 0. \quad (9)$$

This motivates a simple computational method for generating the boundary of $\sigma(\mathcal{P})$. In Section IV, we show that the robust stability results in [2]–[4] on spherical families of polynomials "fall out" quite naturally from the $\Phi(z)$ characterization of $\sigma(\mathcal{P})$.

The Polytopic Problem: In this case, $\mathcal{P}$ is a so-called *polytopic family* described by

$$\mathcal{P} = \text{conv} \{p_1(s), p_2(s), \dots, p_m(s)\} \quad (10)$$

where the $p_i(s)$ are fixed n th-order polynomials. This means that every polynomial $p(s)$ in $\mathcal{P}$ is a convex combination of the form

$$p(s) = \sum_{i=1}^m \lambda_i p_i(s) \quad (11)$$

where $\lambda_i \geq 0$ for $i = 1, 2, \dots, m$ and

$$\sum_{i=1}^m \lambda_i = 1. \quad (12)$$

Again, all polynomials in $\mathcal{P}$ are assumed to have the degree equal to n . Letting a_{ni} denote the coefficient of s^n in $p_i(s)$, note that this degree requirement is satisfied if and only if all the a_{ni} have the same sign.

Remarks: For this polytopic case, we will present a computationally tractable method to compute $\sigma(\mathcal{P})$ which avoids the combinatoric explosion associated with an m -dimensional gridding of the λ set. Instead, we only require a two-dimensional gridding of a bounded subset of the complex plane.

III. THE SPHERICAL PROBLEM

Given a *candidate* point z in the complex plane, it follows that $z \in \sigma(\mathcal{P})$ if and only if

$$A(z)q = - \begin{bmatrix} \text{Re } p_0(z) \\ \text{Im } p_0(z) \end{bmatrix} \quad (13)$$

for some $q \in Q$ where

$$A(z) \doteq \begin{bmatrix} 1 & \text{Re } z & \dots & \text{Re } z^n \\ 0 & \text{Im } z & \dots & \text{Im } z^n \end{bmatrix}. \quad (14)$$

Equivalently, letting

$$\vec{p}_0(z) \doteq \begin{bmatrix} \text{Re } p_0(z) \\ \text{Im } p_0(z) \end{bmatrix}, \quad (15)$$

$z \in \sigma(\mathcal{P})$ exists if and only if the minimum norm solution $q(z)$ of (13) satisfies

$$q^T(z)q(z) \leq r^2. \quad (16)$$

We consider two cases.

Case 1: $\text{Im } z \neq 0$. Then

$$\text{rank } A(z) = 2 \quad (17)$$

and the minimum norm solution of (13) is given by

$$q(z) = -A^T(z)(A(z)A^T(z))^{-1}\vec{p}_0(z). \quad (18)$$

Hence, using (16), $z \in \sigma(\mathcal{P})$ if and only if

$$\bar{p}_0^T(z) (A(z) A^T(z))^{-1} \bar{p}_0(z) \leq r^2. \quad (19)$$

Note that the *weighting matrix*

$$W(z) \doteq (A(z) A^T(z))^{-1} \quad (20)$$

above is *universal* in the sense that is independent of the nominal polynomial $p_0(s)$ —it depends only on the degree n . In fact, since $A(z) A^T(z)$ is a 2×2 matrix, $W(z)$ is easily computed in closed form. Indeed, a straightforward computation leads to the formula

$$W(z) = \frac{1}{\Delta(z)} \begin{bmatrix} \sum_{i=1}^n (\text{Im } z^i)^2 & -\sum_{i=1}^n (\text{Re } z^i)(\text{Im } z^i) \\ -\sum_{i=1}^n (\text{Re } z^i)(\text{Im } z^i) & \sum_{i=0}^n (\text{Re } z^i)^2 \end{bmatrix} \quad (21)$$

where

$$\Delta(z) \doteq \sum_{i=0}^n (\text{Re } z^i)^2 \cdot \sum_{i=1}^n (\text{Im } z^i)^2 - \left(\sum_{i=1}^n (\text{Re } z^i)(\text{Im } z^i) \right)^2. \quad (22)$$

Case 2: $\text{Im } z = 0$. Now, noting that $\bar{p}_0(z)$ is real and $\text{Re } z^i = z^i$ for $i = 0, 1, 2, \dots, n$ and $\text{Im } z^i = 0$ for $i = 1, 2, \dots, n$, (14) becomes

$$\begin{bmatrix} 1 & z & z^2 & \cdots & z^n \end{bmatrix} q = p_0(z). \quad (23)$$

In this case, the minimum norm solution is

$$q(z) = -\frac{1}{\sum_{i=0}^n z^{2i}} \begin{bmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^n \end{bmatrix} p_0(z) \quad (24)$$

and by (16), it follows that $z \in \sigma(\mathcal{P})$ if and only if

$$p_0^2(z) - r^2 \sum_{i=0}^n z^{2i} \leq 0. \quad (25)$$

Therefore, the real roots of the polynomial on the left-hand side of (25) describe the endpoints of intervals where $\sigma(\mathcal{P})$ intersects the real axis.

In conclusion, for both cases, $z \in \sigma(\mathcal{P})$ belongs to the spectral set if and only if an inequality of the form $\Phi(z) \leq 0$ is satisfied, where

$$\Phi(z) \doteq \begin{cases} \bar{p}_0^T(z) W(z) p_0(z) - r^2 & \text{if } \text{Im } z \neq 0; \\ p_0^2(z) - r^2 \sum_{i=0}^n z^{2i} & \text{if } \text{Im } z = 0. \end{cases} \quad (26)$$

Computational Generation of the Spectral Set: Since all points z in the boundary of the spectral set satisfy the simple equality $\Phi(z) = 0$, there are many computational methods available to generate $\sigma(\mathcal{P})$. For example, using 3-D graphics, one can simply plot $\Phi(z)$ and take a cross section at $\Phi(z) = 0$. Another possibility is to “track” the boundary $\sigma(\mathcal{P})$ in the complex plane; e.g., say z_t is the current iterate on the boundary of $\sigma(\mathcal{P})$ and $\text{Im } z_t \neq 0$. Let

$$d^t \doteq \begin{bmatrix} d_1^t \\ d_2^t \end{bmatrix} \quad (27)$$

represent the step direction in the complex plane; i.e., d_1^t represents the real part and d_2^t represents the imaginary part. Then enforcing the condition

$$\Phi(z_{t+1}) = 0 \quad (28)$$

and using a first-order approximation to changes in $\Phi(z)$, we obtain

$$d^t = \left\{ \left(\frac{\partial \Phi(z)}{\partial \text{Re } z} \Big|_{z_t} \right)^2 + \left(\frac{\partial \Phi(z)}{\partial \text{Im } z} \Big|_{z_t} \right)^2 \right\}^{-1/2} \begin{bmatrix} -\frac{\partial \Phi(z)}{\partial \text{Im } z} \Big|_{z_t} \\ \frac{\partial \Phi(z)}{\partial \text{Re } z} \Big|_{z_t} \end{bmatrix} \quad (29)$$

at all points where at least one of the partial derivatives of $\Phi(z)$ is nonvanishing.

Note that the required partial derivatives above can either be supplied in closed form using the formula for $W(z)$ or approximated using some technique such as finite differences. To avoid buildup of numerical error, it is also worthwhile to project z_{t+1} onto the boundary of the set described by $\Phi(z) = 0$. This is accomplished via a simple line search; e.g., find $\lambda \in \mathbf{R}$ such that $\Phi(\lambda z_{t+1}) = 0$.

IV. RELATION WITH ROBUST STABILITY LITERATURE

Recalling Section I, the robust stability problem can be viewed as a special case of the spectral set problem. That is, robust stability is guaranteed if and only if the spectral set does not cross the imaginary axis. Hence, to specialize the results above, we take $z = j\omega$. Then, for $\omega \neq 0$, the formulas describing $\sigma(\mathcal{P})$ greatly simplify. Namely, a straightforward calculation leads to the weighting matrix

$$W(j\omega) = \begin{bmatrix} (\sum_{i \text{ even}} \omega^{2i})^{-1} & 0 \\ 0 & (\sum_{i \text{ odd}} \omega^{2i})^{-1} \end{bmatrix}. \quad (30)$$

Now, using (26), it follows that if $\omega \neq 0$, then $j\omega \in \sigma(\mathcal{P})$ if and only if

$$\frac{(\text{Re } p(j\omega, 0))^2}{\sum_{i \text{ even}} \omega^{2i}} + \frac{(\text{Im } p(j\omega, 0))^2}{\sum_{i \text{ odd}} \omega^{2i}} \leq r^2. \quad (31)$$

For $\omega = 0$, the condition $0 \in \sigma(\mathcal{P})$ becomes

$$|p_0(0)| \leq r. \quad (32)$$

Equivalently, if a_0 denotes the zeroth-order coefficient of $p_0(0)$, then (32) becomes

$$|a_0| \leq r. \quad (33)$$

In conclusion, robust stability is guaranteed if and only if

$$|a_0| > r \quad (34)$$

and

$$\frac{(\text{Re } p(j\omega, 0))^2}{\sum_{i \text{ even}} \omega^{2i}} + \frac{(\text{Im } p(j\omega, 0))^2}{\sum_{i \text{ odd}} \omega^{2i}} > r^2 \quad (35)$$

for all $\omega \in \mathbf{R}$. These conditions amount to a simplification of the results in [2] and are precisely the type of conditions obtained in [3] and [4].

V. THE POLYTOPIC PROBLEM

In this section, we provide a simple technique for spectral set generation which avoids the combinatoric explosion discussed in Section I when the family $\mathcal{P}$ is polytopic. Since this technique really amounts to a specialization of the so-called robust root locus in [5], the exposition is greatly abbreviated. Indeed, given the polytopic family $\mathcal{P}$ and some fixed $z \in \mathbf{C}$, the so-called value set is given by

$$\Omega(z) \doteq \{p(z) : p(\cdot) \in \mathcal{P}\}. \quad (36)$$

In view of the polytopic description of $\mathcal{P}$, we have

$$\Omega(z) = \text{conv} \{p_1(z), p_2(z), \dots, p_m(z)\} \quad (37)$$

Fig. 1. Spectral set for a spherical family.

Fig. 2. Spectral set for a polytopic (interval polynomial) family.

and it follows that

$$\sigma(\mathcal{P}) = \{z \in \mathbb{C} : 0 \in \Omega(z)\}. \quad (38)$$

This suggests the algorithm below for generation of $\sigma(\mathcal{P})$.

Step 1: Compute some *a priori* upper bound $\sigma^+(\mathcal{P})$ (perhaps crude) for $\sigma(\mathcal{P})$. For example, writing

$$p_j(s) = \sum_{i=0}^n a_{ij}s^i \quad (39)$$

for $j = 1, 2, \dots, n$, one such bound given in [6] is

$$\sigma^+(\mathcal{P}) = \{s : |s| \leq R\} \quad (40)$$

where

$$R = 1 + \frac{\max \{a_{ij} : i = 0, 1, \dots, n-1; j = 1, 2, \dots, m\}}{\min \{a_{nj} : j = 1, 2, \dots, m\}}. \quad (41)$$

Step 2: Generate a suitably fine gridding of $\sigma^+(\mathcal{P})$.

Step 3: For each grid point $z \in \sigma^+(\mathcal{P})$, check if $0 \in \Omega(z)$. If $0 \in \Omega(z)$, then we classify z as a point in $\sigma(\mathcal{P})$. If not, z is rejected. The net result of the procedure described above is a set of "clouds" representing $\sigma(\mathcal{P})$. Notice that checking the condition $0 \in \Omega(z)$ is quite simple because $\Omega(z)$ is a two-dimensional convex polygon. This procedure is easily automated as illustrated in the example below. It is also worth noting that the problem of checking whether a point lies in a two-dimensional polygon is well known in computational geometry (e.g., see [7]).

VI. ILLUSTRATIVE EXAMPLES

For the case of a spherical family of polynomials with nominal

$$p_0(s) = s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 6s + 7 \quad (42)$$

and $r = 0.99$, the spectral set was computed using a boundary tracking algorithm as described in Section III. The results are shown in Fig. 1.

For the polytopic case, we take an example considered previously in [8] in the context of robust stability of interval polynomials with respect to the unit circle. Here, however, we generate the entire spectral set. Indeed, we consider a so-called interval polynomial described by

$$p(s, q) = (5 + q_4)s^4 + (1 + q_3)s^3 + (1 + q_2)s^2 + (1 + q_1)s + (1 + q_0) \quad (43)$$

and $|q_i| \leq 0.85$ for $i = 0, 1, 2, 3, 4$. This leads to a polytope of polynomials $\mathcal{P}$ with 32 generators $p_i(s)$ obtained using the extreme values of the q_i . Then, using the two-dimensional gridding technique described in Section V, we generate the spectral set in Fig. 2.

VII. CONCLUSION

In this note, we provided methods for spectral set generation for both spherical and polytopic families of polynomials. In the spherical case, a closed form solution was exploited; in the polytopic case, the computational method involved a two-dimensional gridding of a bounded subset of the complex plane.

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REFERENCES

- [1] B. R. Barmish, "New tools for robustness analysis," in *Proc. IEEE Conf. Decision Contr.*, Austin, TX, 1988.
- [2] C. B. Soh, C. S. Berger, and K. P. Dabke, "On the stability properties of polynomials with perturbed coefficients," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1033-1036, 1985.
- [3] S. P. Bhattacharyya and H. Chapellat, "Stability margin for Hurwitz polynomial," in *Proc. IEEE Conf. Decision Contr.*, Austin, TX, 1988.
- [4] D. Hinrichsen and A. S. Pritchard, "New robustness results for linear systems under real perturbations," in *Proc. IEEE Conf. Decision Contr.*, Austin, TX, 1988.
- [5] B. R. Barmish and R. Tempo, "The robust root locus," *Automatica*, vol. 26, pp. 283-292, 1990.

- [6] M. Marden, "Geometry of Polynomials," in *Mathematical Surveys*, no. 3. Providence, RI: American Mathematical Society, 1966.
- [7] F. P. Preparata and M. I. Shamos, *Computational Geometry*. New York: Springer-Verlag, 1985.
- [8] N. K. Bose, E. Jury, and E. Zeheb, "On robust Hurwitz and Schur polynomials," in *Proc. IEEE Conf. Decision Contr.*, Athens, Greece, 1987.

Discrete-Time Filtering of Noise Correlated Continuous-Time Processes: Modeling and Derivation of the Sampling Period Sensitivities

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Abstract—The optimal discrete-time state estimation of continuous-time processes whose measurements are corrupted by additive white noise is considered in the case where the measurements are prefiltered by an integrator between sampling times. A discrete-time equivalent model, in which the measurements are written as a function of the state vector at the same instant, is developed for the general case where the continuous-time measurement and process noise signals are correlated. The equations governing the optimal filter, which is based on the discrete-time equivalent model, are presented. The properties of this filter are investigated, in the case of a short sampling period, by deriving the first coefficients of the Maclaurin's expansions of the optimal gain and the error covariance matrices in powers of the sampling period. The obtained results are compared to the corresponding expressions that have been previously derived for the sampled-data regulator.

I. INTRODUCTION

In the present note, we deal with the discrete-time optimal estimation of continuous-time processes whose measurements are corrupted by an additive white measurement noise. Due to the presence of the white noise, the measurement signal cannot be directly sampled, and a prefilter must be placed before the sampler [1]. A prefilter, which is commonly implemented, applies an analog integration on the measurement signal, which is reset to zero at the beginning of each sampling interval [1]–[7]. The prefilter, which becomes a part of the overall continuous-time system, is considered in the present note.

In the case where the continuous-time measurement and the process noise signals are uncorrelated, the discrete-time optimal estimator is obtained by applying a variant of the discrete-time Kalman filter algorithm on the discrete-time equivalent model (DTEM) of the overall sampled continuous-time system, e.g., [1], [6], and [7].

A DTEM for the case where the measurement and the input noise signals of the continuous-time system are correlated has been previously obtained in [1]. This model is characterized by the fact that the measurements at the sampling time t_k are expressed as a function of the state vector at time t_{k-1} . Models with this characteristic are not commonly used, since most of the results in discrete-time state estimation refer to discrete-time models where the measurement vector at time t_k is a function of the state vector at the same instant. In the present note, we develop a DTEM with this characteristic for the general correlated noise signals case. Based on this

exact model, we derive recursive equations of the optimal discrete-time filter. These equations do not provide, however, an easy way of determining how the parameters of the system affect the achievable error covariance matrices and the required sampling period. A good insight into the role of these parameters can be gained in the stationary case from the calculation of the sampling period sensitivities, i.e., the first coefficients of the Maclaurin's expansions in the small sampling period of the gain and the error covariance matrices. By means of these sensitivities, we can easily determine, as accurately as desired, the gain and the error covariance matrices for short enough sampling periods.

These lines of analysis have been previously applied to the solution of the regulator problem, e.g., [8]–[10]. The reason they have not been adopted so far for the solution of the filtering problem is probably the well-known duality between the linear filtering and regulation problems, both in the continuous- and discrete-time cases. It was probably taken for granted that the results that had been obtained in [9] and [10] for the regulation problem are valid, by duality considerations, also for the filtering case. It has been shown, however, in [6] that the duality is not preserved after discretization when an integrator between sampling times is used as the prefilter in the filtering case, and a sample-and-hold technique is implemented in the regulation problem. The results of [9] and [10] can not, therefore, be used directly in the filtering problem.

In [7], the first coefficient of the Maclaurin's expansion of the *a priori* stationary error covariance matrix is calculated for the uncorrelated noise signals case. In the present note, we extend this result by calculating the first three sampling period sensitivities (SPS's) of the gain, the *a priori* and the *a posteriori* error covariance matrices, for the general stationary case where the noise signals are correlated. As a particular case, we treat the uncorrelated noise signals case where we compare the derived SPS's with the already known SPS's of the sampled-data regulator. This note is organized as follows. The problem is formulated in Section II. The DTEM and the discrete-time filter are derived in Section III. In Section IV, the SPS's of the stationary gain and the corresponding error covariance matrices are calculated and used for determining the required sampling period, and in Section V they are compared to the SPS's that have been obtained for the sampled-data regulator.

II. PROBLEM FORMULATION

We consider the following stabilizable and detectable representation of a continuous-time system:

$$\dot{x}(t) = A(t)x(t) + w(t), \quad (2.1a)$$

$$y(t) = C(t)x(t) + v(t) \quad (2.1b)$$

where $w(t)$ and $v(t)$ are zero-mean white noise signals that satisfy

$$E\{w(t)w'(\tau)\} = Q(t)\delta(t-\tau), \quad (2.2a)$$

$$E\{v(t)v'(\tau)\} = R(t)\delta(t-\tau), \quad (2.2b)$$

and

$$E\{w(t)v'(\tau)\} = M(t)\delta(t-\tau) \quad (2.2c)$$

where $Q \geq 0$ and $R > 0$. Denoting the sampling period by $T \triangleq t_k - t_{k-1}$, the configuration shown in Fig. 1 is considered.

III. THE DISCRETE-TIME EQUATIONS

A. The Discrete-Time Equivalent Model (DTEM)

By using a similar procedure to the one used in [3] and [6], for the case where the original continuous-time process and the measurement noise signals are uncorrelated, we obtain a DTEM for the combination of the continuous-time process and the prefilter, in the case where the continuous-time process and the measurement noise

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