REFERENCES

- [1] R. W. Brockett, "Least square matching problems," *Linear Algebra its Appl.*, vol. 122/123/124, pp. 761-777, 1989.
- [2] —, "Dynamic systems that sort lists, diagonalize matrices and solve linear programming problems," *Linear Algebra its Appl.*, vol. 146, pp. 79-91, 1991.
- [3] J. Aloimonos and I. Rigoutsos, "Determining 3-D motion of a surface patch without correspondences under perspective projection," in *Proc. AAAI*, Aug. 1986, pp. 681-688.
- [4] H. Lee, Z. Lin, and T. S. Huang, "Finding 3-D point correspondences in motion estimation," in *Proc. Eighth Int. Conf. Pattern Recognition*, 1986, pp. 303-305.
- [5] D. B. Goldgof, H. Lee, and T. S. Huang, "Matching and motion estimation of three dimensional point and line sets using eigenstructure without correspondences," *Pattern Recognition*, vol. 25, no. 3, pp. 271-286, 1992.
- [6] U. Helmke and J. B. Moore, *Optimization and Dynamical Systems*. New York: Springer-Verlag, 1994.

Discontinuous Perturbation Analysis of Discrete-Event Dynamic Systems

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Abstract—The discontinuous perturbation analysis (DPA) method is presented to deal with discontinuous sample performance functions. This method is applicable to derivative estimation with respect to most threshold-type problems in discrete-event dynamic systems (DEDS). Through modeling of discontinuities by step function, we are able to provide a unified framework for constructing the derivative estimation of a DEDS. As a result, we offer an alternate approach to derive derivative estimation in DEDS's. Finally, some relationships with other PA techniques are discussed, and a numerical example is presented.

I. INTRODUCTION

Consider a sample path of stochastic discrete-event dynamic systems (DEDS) represented by a pair (θ, ω) , where θ is a decision parameter and ω is a sequence of random numbers, defined on a probability space $(\Omega, \mathcal{F}, P)$, which represents the random effects involved in the evolution of the system. In computer simulation, ω is a sequence of uniformly distributed random numbers on $[0,1]$ from which we can generate other arbitrarily distributed random numbers. Let $L(\theta, \omega)$ be a performance measure of the system. Often, we are interested in derivatives of expected performance $E[L(\theta, \omega)]$, e.g., for optimization or sensitivity analysis. One means of deriving estimates for such derivatives from a sample path (or simulation) of the system is infinitesimal perturbation analysis (IPA) [13], [3]. Another technique available for the derivatives estimate of expected performance $E[L(\theta, \omega)]$ is the score function method (SF) or likelihood ratio method (LR) [11], [16], [15]. The domains of

problems with which IPA and LR derivative estimates can be used overlap, but neither contains the other [14].

Many experiments have shown that IPA, when applicable, produces a smaller variance of estimation error (based on fixed simulation length) than others [8]. However, IPA fails for most threshold (or structure) types of problems, such as the routing problem and queue length. The reason is mainly due to the sample path discontinuity with respect to the parameter in question, i.e., infinitesimal change in parameter value produces finite change in the sample path of the system. There has been a considerable effort to extend the theory of the IPA to handle the situation where IPA fails [1], [5], [12], [18], [4].

In this paper, we propose another alternative approach called a discontinuous perturbation analysis (DPA) which tries to deal with discontinuity of sample performance functions, especially for threshold types of parameters or structure parameters. In this approach we use the step function to represent a discontinued sample performance function. We then introduce the function to the sample path derivative. With this format, we are able to provide a framework for constructing the derivative estimation of certain classes of DEDS's. We will show that the DPA formulation derived from this approach is very closely related to other exiting methods such as smoothed perturbation analysis (SPA) [2], rare perturbation analysis (RPA) [1], and LR. In some cases, the SPA formula can be derived from the DPA formula. Variance comparisons between DPA and other PA techniques will be discussed.

The remainder of the paper is organized as follows. The discontinuity of the sample performance function is represented by using step functions in Section II. In Section III, we will introduce the δ -function in the sample path derivative. We then derive the DPA formula from this new setting. In Section IV, we will explore the relationship between the DPA and other PA techniques. Variance comparison between DPA and other PA techniques is presented. Conclusions are drawn in Section V.

II. PROBLEM FORMULATION

We start to formulate the problem through an example. Let us consider a $G/G/1$ queue with the service time

$$S(\theta, \xi) = \begin{cases} S^+(\theta, \xi) & \text{if } 0 \leq \xi \leq \theta, \\ S^-(\theta, \xi) & \text{if } 1 \geq \xi > \theta. \end{cases} \quad (1)$$

Assume that the performance function we are interested in is mean waiting time. When $S^+(\theta, \xi) \neq S^-(\theta, \xi)$, $S(\theta, \xi)$ is discontinuous with respect to θ . Generally, in this case, IPA will not work. To be able to derive the derivative estimator for this problem, we use a fundamentally different way to formulate the problem. Suppose the simulation run terminates at the n th arrival. Let $\sigma(\xi) = (\sigma(\xi_1), \dots, \sigma(\xi_n))$ be an n -dimensional vector such that

$$\sigma(\xi_i) = \begin{cases} 0 & \text{if } 0 \leq \xi_i \leq \theta \text{ (service time is } S^+), \\ 1 & \text{if } \theta < \xi_i \leq 1 \text{ (service time is } S^-). \end{cases} \quad (2)$$

Using this notation, given a sample realization $\xi = (\xi_1, \dots, \xi_n)$, we can determine $\sigma(\xi)$ and the corresponding service time. We denote the interarrival time by the symbol η . Here η is a vector and is independent of $\sigma(\xi)$. The sample performance function then can be written as follows:

$$L(\theta, \omega, n) \equiv f(\theta, \xi, \eta, \sigma(\xi)). \quad (3)$$

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Denote L_{+i} and L_{-i} as follows:

$$\begin{aligned} L_{+i} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=0} \\ &\text{(i.e., the } i\text{th customer has service time } S^+(\theta, \xi)) \\ L_{-i} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=1} \\ &\text{(i.e., the } i\text{th customer has service time } S^-(\theta, \xi)). \end{aligned}$$

Then given $(\sigma(\xi_1), \dots, \sigma(\xi_n))$, $L(\theta, \omega, n)$ is either equal to L_{+i} (if $\sigma(\xi_i) = 0$) or L_{-i} if $(\sigma(\xi_i) = 1)$. The value $\sigma(\xi_j)$, for $j \neq i$, is the same in $L(\theta, \omega, n)$, L_{+i} , and L_{-i} , i.e., $L(\theta, \omega, n)$ can be represented as, for any $i \leq n$

$$L(\theta, \omega, n) = L_{+i} + (L_{-i} - L_{+i})H(\xi_i - \theta) \quad (4)$$

where $H(\xi - \theta)$ is the step function

$$H(\xi_i - \theta) = \begin{cases} 0 & \text{if } 0 \leq \xi_i \leq \theta, \\ 1 & \text{if } \theta < \xi_i \leq 1. \end{cases} \quad (5)$$

In a similar vein, we can define

$$\begin{aligned} L_{+i,+j} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=0, \sigma(\xi_j)=0} \\ L_{+i,-j} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=0, \sigma(\xi_j)=1} \\ L_{-i,+j} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=1, \sigma(\xi_j)=0} \\ L_{-i,-j} &= f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=1, \sigma(\xi_j)=1}. \end{aligned}$$

Hence each L_{+i} and L_{-i} can be represented as

$$L_{+i} = L_{+i,+j} + (L_{+i,-j} - L_{+i,+j})H(\xi_j - \theta) \quad (6)$$

$$L_{-i} = L_{-i,+j} + (L_{-i,-j} - L_{-i,+j})H(\xi_j - \theta). \quad (7)$$

Note that for this problem, the dependence of the performance function on the parameter θ is not only through $S^+(\theta, \xi)$ (or $S^-(\theta, \xi)$) but also through $\sigma(\xi)$ as well. The IPA derivative estimator can only reflect the dependence of the performance function on the parameter θ through $S^+(\theta, \xi)$ (or $S^-(\theta, \xi)$) only, but not $\sigma(\xi)$. In the following section, we will show that using a step function in sample path performance, we are able to capture the dependence on the parameter θ through $\sigma(\xi)$ as well.

In this paper, sample performance function (3) is considered. Note that many performance functions for threshold types of problems in DEDS can be represented in (3), such as routing problems.

III. DISCONTINUOUS PERTURBATION ANALYSIS

In the literature [7, p. 4] $H(\xi - \theta)$ is called a generalized function whose derivatives are also generalized functions. Before giving the main results, let us first introduce a lemma.

Lemma: Assume that $k(\theta, \xi)$ satisfies the following conditions: i) $k(\theta, \xi)$ is uniformly differentiable with respect to θ and ii) $k(\theta, \xi)$ is continuous for any given θ . Let

$$J(\theta) = \int_0^1 k(\theta, \xi)H(\xi - \theta) d\xi$$

then

$$\begin{aligned} \partial J(\theta)/\partial \theta &= \int_0^1 \{k'_\theta(\theta, \xi)H(\xi - \theta) - k(\theta, \xi)\delta(\xi - \theta)\} d\xi \\ &= \int_0^1 k'_\theta(\theta, \xi)H(\xi - \theta) d\xi - k(\theta, \theta) \end{aligned} \quad (8)$$

where $k_\theta = \partial k / \partial \theta$. The Dirac δ -function, $\delta(\xi - \theta)$, satisfies, for any continuous function $k(\xi)$

$$\int_0^1 k(\xi)\delta(\xi - \theta) d\xi = k(\theta). \quad (9)$$

The lemma can be verified by taking the derivative to the following equation:

$$J(\theta) = \int_0^1 k(\theta, \xi)H(\xi - \theta) d\xi = \int_0^1 k(\theta, \xi) d\xi.$$

Remark 1: Equation (8) contains two terms. The first term corresponds to the IPA estimate. The second term is an expectation conditioned on the event $\xi = \theta$ since the expression $\delta(\xi - \theta) d\xi$ corresponds to the density of the probability distribution that is concentrated at the point θ . The second term clearly explains that the presence of discontinuities related to events that have a positive probability of occurring ($k(\theta, \theta) \neq 0$) causes the failure of the IPA algorithm. The same observation can also be found in [6] and [9].

Theorem 1: Assume that the sample performance function $L(\theta, \omega, n)$ satisfies (2) and (3). Given $\sigma(\xi)$, $f(\theta, \xi, \eta, \sigma(\xi))$ satisfies:

- 1) $f(\theta, \xi, \eta, \sigma(\xi))$ is uniformly differentiable with respect to θ ;
- 2) $f(\theta, \xi, \eta, \sigma(\xi))$ is a continuous function of ξ .

Then

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L(\theta, \omega, n)] &= E \left[\frac{\partial}{\partial \theta} L(\theta, \omega, n) |_{\text{IPA}} \right] \\ &\quad + E \left[\sum_{i=1}^n (L_{+i\theta} - L_{-i\theta}) \right] \end{aligned} \quad (10)$$

where $(\partial/\partial \theta)L(\theta, \omega, n)|_{\text{IPA}} = (\partial/\partial \theta)f(\cdot, \xi, \eta, \sigma(\xi))$

$$L_{+i\theta} = f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=0, \xi_i=\theta}, \quad \text{and}$$

$$L_{-i\theta} = f(\theta, \xi, \eta, \sigma(\xi))|_{\sigma(\xi_i)=1, \xi_i=\theta}.$$

Proof: We argue by induction. When $n = 1$ from (4), we know

$$L(\theta, \omega, 1) = L_{+1} + (L_{-1} - L_{+1})H(\xi_1 - \theta).$$

From conditions i) and ii) and the lemma, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L(\theta, \omega, 1)] &= \frac{\partial}{\partial \theta} \left\{ \int_0^1 E_\eta[L_{+1}] d\xi_1 + \int_0^1 (E_\eta[L_{-1}] \right. \\ &\quad \left. - E_\eta[L_{+1}])H(\xi_1 - \theta) d\xi_1 \right\} \\ &= \int_0^1 E_\eta \left[\frac{\partial}{\partial \theta} L_{+1} \right] d\xi_1 + \int_0^1 \left(E_\eta \left[\frac{\partial}{\partial \theta} L_{-1} \right] \right. \\ &\quad \left. - E_\eta \left[\frac{\partial}{\partial \theta} L_{+1} \right] \right) H(\xi_1 - \theta) d\xi_1 \\ &\quad + \int_0^1 (E_\eta[L_{+1}] - E_\eta[L_{-1}])\delta(\xi_1 - \theta) d\xi_1 \\ &= \int_0^1 E_\eta \left[\frac{\partial}{\partial \theta} L(\theta, \omega, 1) |_{\text{IPA}} \right] d\xi_1 \\ &\quad + (E_\eta[L_{+1}(\xi_1 = \theta)] - E_\eta[L_{-1}(\xi_1 = \theta)]) \\ &= E \left[\frac{\partial}{\partial \theta} L(\theta, \omega, 1) |_{\text{IPA}} \right] + E[L_{+1\theta} - L_{-1\theta}]. \end{aligned}$$

Now suppose that $n = k$, and take as an induction hypothesis that (10) holds for $n = k - 1$, from (4) we have

$$L(\theta, \omega, k) = L_{+k} + (L_{-k} - L_{+k})H(\xi_k - \theta)$$

then

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L(\theta, \omega, k)] &= \frac{\partial}{\partial \theta} \left\{ \int_0^1 E[L_{+k}] d\xi_k + \int_0^1 (E[L_{-k}] \right. \\ &\quad \left. - E[L_{+k}])H(\xi_k - \theta) d\xi_k \right\} \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 E \left[\frac{\partial}{\partial \theta} L_{+k} \right] d\xi_k + \int_0^1 \left(E \left[\frac{\partial}{\partial \theta} L_{-k} \right] \right. \\
&\quad \left. - E \left[\frac{\partial}{\partial \theta} L_{+k} \right] \right) H(\xi_k - \theta) d\xi_k \\
&\quad + \int_0^1 (E[L_{+k}] - E[L_{-k}]) \delta(\xi_k - \theta) d\xi_k \\
&= \int_0^1 E \left[\frac{\partial}{\partial \theta} L_{+k} \right] d\xi_k + \int_0^1 \left(E \left[\frac{\partial}{\partial \theta} L_{-k} \right] \right. \\
&\quad \left. - E \left[\frac{\partial}{\partial \theta} L_{+k} \right] \right) H(\xi_k - \theta) d\xi_k \\
&\quad + E[L_{+k\theta} - L_{-k\theta}]. \tag{11}
\end{aligned}$$

Because of the hypothesis we have (12), shown at the bottom of the page.

Note that from (6) and (7), L , $L_{+k\theta}$ and $L_{-k\theta}$ can be represented as

$$\begin{aligned}
L(\theta, \omega, k) &= L_{+k} + (k_{-k} - L_{+k})H(\xi_k - \theta) \\
L_{+k\theta} &= L_{+k, +i\theta} + (L_{-k, +i\theta} - L_{+k, +i\theta})H(\xi_k - \theta) \\
L_{-k\theta} &= L_{+k, -i\theta} + (L_{-k, -i\theta} - L_{+k, -i\theta})H(\xi_k - \theta).
\end{aligned}$$

Combining (11) and (12), we proved the theorem. $\square$

Remark 2: When we use the step function to represent a discontinuous sample performance function, we can obtain the sample path derivative which includes δ -functions. To be able to calculate the sample path derivative, we need to remove all the δ -functions in the sample path derivative. Since each ξ_i is independent of other random variables, we can make an expectation on each ξ_i . Therefore, the δ -function (related to ξ_i) will be removed. By removing the δ -function, we "smooth" the sample path derivative, hence we can still calculate the sample path derivative based on a single run.

We note that the above results can be extended to multithreshold cases with more general setting. Let

$$\begin{aligned}
L(\theta, \omega, n) &= L_i^k(\theta, \xi, \eta), \text{ if } g^{k-1}(\theta) < \xi_i < g^k(\theta), \\
&\quad k = 1, \dots, M, \quad \text{and} \\
0 &= g^0(\theta) < g^1(\theta) < \dots < g^M(\theta) = 1
\end{aligned}$$

then $L(\theta, \omega, n)$ can be represented as

$$L(\theta, \omega, n) = L_i^1 + \sum_{k=1}^{M-1} (L_i^{k+1} - L_i^k) H(\xi_i - g_i^k(\theta)). \tag{13}$$

For example, when $M = 2$, (13) becomes (4), i.e., $L_i^1 = L_{+i}$, and $L_i^2 = L_{-i}$.

For discontinuous sample performance (13), we have the following result.

Theorem 2: Assume that for all i and k , $L_i^k(\theta, \xi, \eta)$ satisfy the conditions of Theorem 1; then we have

$$\begin{aligned}
\frac{\partial}{\partial \theta} E[L(\theta, \omega, n)] &= E \left[\frac{\partial}{\partial \theta} L(\theta, \omega, n) |_{\text{IPA}} \right] \\
&\quad + E \left[\sum_{i=1}^n \sum_{k=1}^M (L_{i\theta}^k - L_{i\theta}^{k+1}) \frac{dg_i^k(\theta)}{d\theta} \right] \tag{14}
\end{aligned}$$

where $L_{i\theta}^k = L_i^k(\xi_i = g_i^k(\theta))$.

Proof: The proof of Theorem 3 is similar to the proof of Theorem 1.

Remark 3: In this paper $g_i(\theta)$ is considered only as a function of θ . Actually, $g_i(\theta)$ can represent the inverse probability distribution of a random variable, since the random variable usually can be generated from the uniformly distributed random variable ξ_i .

$$\begin{aligned}
&\int_0^1 E \left[\frac{\partial}{\partial \theta} L_{+k} \right] d\xi_k + \int_0^1 \left(E \left[\frac{\partial}{\partial \theta} L_{-k} \right] - E \left[\frac{\partial}{\partial \theta} L_{+k} \right] \right) H(\xi_k - \theta) d\xi_k \\
&= \int_0^1 \left(E \left[\frac{\partial}{\partial \theta} L_{+k} |_{\text{IPA}} \right] + E \left[\sum_{i=1}^{k-1} (L_{+k, +i\theta} - L_{+k, -i\theta}) \right] \right) d\xi_k \\
&\quad + \int_0^1 \left(\left(E \left[\frac{\partial}{\partial \theta} L_{-k} |_{\text{IPA}} \right] - E \left[\frac{\partial}{\partial \theta} L_{+k} |_{\text{IPA}} \right] \right) \right. \\
&\quad \left. + \left(E \left[\sum_{i=1}^{-1} (L_{-k, +i\theta} - L_{-k, -i\theta}) \right] - E \left[\sum_{i=1}^{k-1} (L_{+k, +i\theta} - L_{+k, -i\theta}) \right] \right) \right) H(\xi_k - \theta) d\xi_k \\
&= \int_0^1 E \left[\frac{\partial}{\partial \theta} L_{+k} |_{\text{IPA}} \right] + \left(E \left[\frac{\partial}{\partial \theta} L_{-k} |_{\text{IPA}} \right] - E \left[\frac{\partial}{\partial \theta} L_{+k} |_{\text{IPA}} \right] \right) H(\xi_k - \theta) d\xi_k \\
&\quad + \int_0^1 \left(E \left[\sum_{i=1}^{k-1} (L_{+k, +i\theta} - L_{+k, -i\theta}) \right] \right. \\
&\quad \left. + \left(E \left[\sum_{i=1}^{k-1} (L_{-k, +i\theta} - L_{-k, -i\theta}) \right] - E \left[\sum_{i=1}^{k-1} (L_{+k, +i\theta} - L_{+k, -i\theta}) \right] \right) H(\xi_k - \theta) \right) d\xi_k \\
&= \int_0^1 E \left[\frac{\partial}{\partial \theta} (L_{+k} + (L_{-k} - L_{+k})H(\xi_k - \theta)) |_{\text{IPA}} \right] d\xi_k \\
&\quad + \int_0^1 \left(E \left[\sum_{i=1}^{k-1} \{L_{+k, +i\theta} + (L_{-k, +i\theta} - L_{+k, +i\theta})H(\xi_k - \theta)\} \right] \right. \\
&\quad \left. - E \left[\sum_{i=1}^{k-1} \{L_{+k, -i\theta} + (L_{-k, -i\theta} - L_{+k, -i\theta})H(\xi_k - \theta)\} \right] \right) d\xi_k \\
&= E \left[\frac{\partial}{\partial \theta} L(\theta, \omega, k) |_{\text{IPA}} \right] + E \left[\sum_{i=1}^{k-1} (L_{+i\theta} - L_{-i\theta}) \right]. \tag{12}
\end{aligned}$$

Remark 4: Equation (14) gives the general DPA formula. As we mentioned, in most cases the IPA cannot be directly applied to the problem dealt with in this paper simply because for most structure parameters the second term of (14) will not be equal to zero. Therefore, here, we provide a way to verify IPA validity, i.e., by showing that the second term in (14) is equal to zero, we can prove that the IPA estimate is valid. There also exist some other ways to check IPA validity such as the augmented infinitesimal perturbation method (APA) [6].

IV. RELATIONS WITH OTHER DERIVATIVE ESTIMATES

In this section we will point out that the DPA is very closely related to other PA techniques. We use the single $G/G/1$ queue with admission control as our example to illustrate their relationships [2]. The system is a single server with a switch that controls the arrival stream to the server. The control rate is θ .

Proposition 1: For the $G/G/1$ queue with the admission control problem, let the performance function be system waiting time, then:

- 1) the DPA formula is the same as the RPA [1] formula;
- 2) the SPA formula can be derived from the DPA formula, and DPA is the conditional expectation of SPA;
- 3) DPA is the conditional expectation of LR.

Proof:

- 1) Since the sample performance function has the form $L(\theta, \zeta, n) \equiv f(\eta, \sigma(\xi))$ where $\zeta = (\eta, \sigma(\xi))$ and $\sigma(\xi)$ satisfies (2), from Theorem 1, we have DPA estimate formula

$$\frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)] = E \left[\frac{\partial}{\partial \theta} L(\theta, \zeta, n) \right] = E \left[\sum_{i=1}^n (L_{+i} - L_{-i}) \right] \quad (15)$$

which is the same as the RPA formula.

- 2) Notice that ξ_i 's are independent, and the dependence of the performance function on the parameter, θ , is only through $\sigma(\xi)$. From (4) we have

$$E[E_{\xi_i}[L(\theta, \zeta, n)]] = \theta E[L_{+i}] + (1 - \theta) E[L_{-i}] \quad (16)$$

i.e.,

$$E[L_{+i}] - E[L_{-i}] - E[L_{-i}] = \frac{1}{\theta} (E[L(\theta, \zeta, n)] - E[L_{-i}]) \quad (17)$$

or

$$E[L_{+i}] - E[L_{-i}] = \frac{1}{1 - \theta} (E[L_{+i}] - E[L(\theta, \zeta, n)]). \quad (18)$$

Plugging (17) and (18) into the DPA formula (15), respectively, we obtain the following SPA derivative estimate formulas:

$$\frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)] = \frac{1}{\theta} E \left[\sum_{i=1}^n (L(\theta, \zeta, n) - L_{-i}) \right] \quad (19)$$

for any $\theta \in (0, 1]$, and

$$\frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)] = \frac{1}{1 - \theta} E \left[\sum_{i=1}^n (L_{+i} - L(\theta, \zeta, n)) \right] \quad (20)$$

for any $\theta \in [0, 1)$.

Equations (19) and (20) are the SPA left- and right- derivative estimate, respectively [2]. Thus, the SPA estimate formulas can be derived from the DPA formula.

In turn, by taking the expectation on ξ_i to the SPA formula (19) or (20), we will obtain the DPA (15).

- 3) For the $G/G/1$ queue with admission control, the LR estimate formula has the form

$$\frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)]|_{LR} = E \left[\sum_{i=1}^n \frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} L(\theta, \zeta, n) \right] \quad (21)$$

where $P_\theta = \theta$ if $0 \leq \xi_i \leq \theta$, or $P_\theta = 1 - \theta$ if $\theta \leq \xi_i \leq 1$, and $dP_\theta = 1$ if $0 \leq \xi_i \leq \theta$, or $dP_\theta = -1$ if $\theta \leq \xi_i \leq 1$.

Because each ξ_i is independent, by taking an expectation on ξ_i we have

$$E_{\xi_i} \left[\frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} L(\theta, \zeta, n) \right] = L_{+i} - L_{-i}. \quad (22)$$

Plugging (22) into (21) we have the DPA formula (15). $\square$

From the proof we can see that the SPA formula can be derived from the DPA formula. Actually, if the inverse process of deriving SPA is used, the DPA formula will be obtained. To derive the SPA estimate formula, we need to first "smooth" (by taking conditional expectation) the sample performance function and then take the derivative of the sample performance function. On the contrary, to derive the DPA formula, we take the sample path derivative first and then "smooth" (by taking an expectation to remove the δ -function) the sample path derivative. These two processes produce a slightly different derivative estimate formula. For the structure or threshold parameters mentioned in this paper, the DPA approach might provide a simpler and more direct way to deal with the problems over the SPA (or RPA).

We also call the reader's attention to the fact that for the $G/G/1$ admission control problem, the SPA can be derived from the LR formula. Since

$$E_{\xi_i} \left[\frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} \right] = 0 \quad (23)$$

for any function $h(\theta, \zeta, n | \xi_i \text{ is given})$, we have

$$E_{\xi_i} \left[\frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} h(\theta, \zeta, n | \xi_i \text{ is given}) \right] = 0. \quad (24)$$

Choose $h(\theta, \zeta, n | \xi_i \text{ is given}) = L_{-i}$; we have¹

$$E_{\xi_i} \left[\frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} L_{-i} \right] = 0.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)]|_{LR} &= E \left[\sum_{i=1}^n \frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} L(\theta, \zeta, n) \right] \\ &\quad - E \left[\sum_{i=1}^n E_{\xi_i} \left[\frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} L_{-i} \right] \right] \\ &= E \left[\sum_{i=1}^n \frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} (L(\theta, \zeta, n) - L_{-i}) \right]. \end{aligned}$$

Note that n is the total number of customers arriving at the system (including accepted and rejected). Let A be the index set that all customers are accepted and R be the index set that all customers are

¹ We can also choose $h(\theta, \zeta, n | \xi_i \text{ is given}) = L_{+i}$ or any other function which is independent of ξ_i . If we choose $h(\theta, \zeta, n | \xi_i \text{ is given}) = L_{+i}$, (20) will be derived from the LR formula.

rejected. Then the above equation can be written as

$$\begin{aligned} & \frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)]|_{LR} \\ &= E \left[\sum_{i \in A} \frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} (L(\theta, \zeta, n) - L_{-i}) \right] \\ &+ E \left[\sum_{i \in R} \frac{dP_\theta(i-1, i)}{P_\theta(i-1, i)} (L(\theta, \zeta, n) - L_{-i}) \right]. \end{aligned}$$

Since for all $i \in R$, $L(\theta, \zeta, n) = L_{-i}$, the second term in the last equation is equal to zero. And for all $i \in A$, $dP_\theta = 1$, $P_\theta = \theta$, therefore, the above equation can be written as (in expectation)

$$\begin{aligned} &= E \left[\sum_{i \in A} \frac{1}{\theta} (L(\theta, \zeta, n) - L_{-i}) \right] \\ &= \frac{1}{\theta} E \left[\sum_{i=1}^m (L(\theta, \zeta, n) - L_{-i}) \right] \\ &= \frac{\partial}{\partial \theta} E[L(\theta, \zeta, n)]|_{SPA} \end{aligned}$$

where m is the total number of customers accepted in the system.

From the fact that the DPA is the conditional expectation of SPA and LR, essentially we can construct more new derivative estimate formulas. This can be done by choosing suitable $h_i(\theta, \xi, \eta, \sigma(\xi), n)$ such that

$$E_{\xi_i}[h_i(\theta, \xi, \eta, \sigma(\xi), n)] = L_{+i\theta} - L_{-i\theta}.$$

For example, for the $G/G/1$ admission control problem, if we take

$$h_i = \frac{dP_\theta}{P_\theta} [\theta(L - L_{-i}) - (1 - \theta)(L_{+i} - L)] \quad (25)$$

then we obtain a new derivative estimate formula (called DPA1)

$$\begin{aligned} \frac{d}{d\theta} E[L]|_{DPA1} &= E \left[\sum_{i=1}^n \frac{dP_\theta}{P_\theta} [(\theta(L - L_{-i}) - (1 - \theta)(L_{+i} - L))] \right] \\ &= E \left[\sum_{i \in A} (L - L_{-i}) \right] + E \left[\sum_{i \in R} (L_{+i} - L) \right]. \quad (26) \end{aligned}$$

We can also choose

$$h_i = \begin{cases} L_{+i} & \text{if } 0 < \xi_i \leq \theta \\ L_{+i} - \frac{1}{1-\theta} L_{-i} & \text{if } \theta < \xi_i \leq 1 \end{cases}$$

then we obtain another new derivative formula (DPA2)

$$\begin{aligned} \frac{d}{d\theta} E[L]|_{DPA2} &= E \left[\sum_{i \in A} L_{+i} \right] \\ &+ E \left[\sum_{i \in R} (L_{+i} - \frac{1}{1-\theta} L_{-i}) \right]. \quad (27) \end{aligned}$$

As we can imagine, many other new derivative estimate formulas can be derived this way. This should not be surprising, since the DPA provides a very general setting for this type of problem.

Now all of these formulas, LR, SPA, DPA, RPA, ... yield unbiased estimates since all of them would compute the quantities under expectation in the right-hand side of their respective definitions. The comparisons must be based on their variance. It is well known that the variance of the LR estimate goes to infinity when n goes to infinity. Usually, using the LR estimate formula we will have the variance problem. In what follows, we present a numerical example to compare the variance of the DPA estimate formula and the SPA formula.

Example: An $M/G/1$ queue is simulated. The service time is given as the follows:

$$S_i = \begin{cases} 0.8 - \xi_i & \text{if } 0 \leq \xi_i \leq \theta \\ \xi_i, & \text{if } 1 \geq \xi_i > \theta \end{cases} \quad (28)$$

where $0 < \theta \leq 0.8$.

For this problem, the DPA estimate formula is

$$\frac{\partial}{\partial \theta} E[L]|_{DPA} = \sum_{i=1}^M (L_{+i\theta} - L_{-i\theta}) \quad (29)$$

where M is the total number of customers simulated and the SPA estimate formula has the following form:

$$\frac{\partial}{\partial \theta} E[L]|_{SPA} = \frac{1}{\theta} \sum_{i \in A} (L - L_{-i\theta}) \quad (30)$$

where $A = \{i: \xi_i > \theta \text{ and } 1 \leq i \leq M\}$.

Remember that $L_{+i\theta}$ is the same as the original sample path L except the i th service time is replaced by $S_{+i\theta} (= 0.8 - \theta)$. Also $L_{-i\theta}$ is the same as the original sample path L except the i th service time is replaced by $S_{-i\theta} (= \theta)$. Let T_k be k th customer system time, T_k^{+i} be the k th customer system time when the original i th customer service time replaced is by $S_{+i\theta}$, and T_k^{-i} be the k th customer system time when the original i th customer service time is replaced by $S_{-i\theta}$, then

$$L_{+i\theta} = \sum_{k=1}^M T_k^{+i} = \sum_{k=1}^M (d^{+i}(k) - a(k)) \quad (31)$$

and

$$L_{-i\theta} = \sum_{k=1}^M T_k^{-i} = \sum_{k=1}^M (d^{-i}(k) - a(k)) \quad (32)$$

where $a(k)$ is the k th customer arrival time, and $d(k)$ is the k th customer departure time, $d^{+i}(k)$ and $d^{-i}(k)$ are the k th customer departure times when the i th customer service time is replaced by $S_{+i\theta}$ and $S_{-i\theta}$, respectively. Substitute (31) and (32) into (30), and we have

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L]|_{DPA} &= \sum_{i=1}^M (L_{+i\theta} - L_{-i\theta}) \\ &= \frac{1}{M} \sum_{i=1}^M \sum_{k=1}^M (T_k^{+i\theta} - T_k^{-i\theta}) \\ &= \frac{1}{M} \sum_{i=1}^M \sum_{k=1}^M (d^{+i}(k) - d^{-i}(k)). \quad (33) \end{aligned}$$

Note that when $k < i$, because the k th customer departure time will not be effected by the change of the i th customer service, $d^{+i}(k) = d^{-i}(k)$. The above equation can be written as

$$\begin{aligned} \frac{\partial}{\partial \theta} E[L]|_{DPA} &= \sum_{i=1}^M (L_{+i\theta} - L_{-i\theta}) \\ &= \frac{1}{M} \sum_{i=1}^M \sum_{k=1}^M (d^{+i}(k) - d^{-i}(k)) \quad (34) \end{aligned}$$

where $d^{+i}(k)$ and $d^{-i}(k)$ can be calculated by using the following forms:

$$d(k) = \max\{a(k) + S_k, d(k-1) + S_k\} \quad (35)$$

$$d^{+i}(i) = \max\{a(k) + S_{+i\theta}, d(k-1) + S_{+i\theta}\} \quad (36)$$

$$d^{+i}(k) = \max\{a(k) + S_k, d^{+i}(k-1) + S_k\} \quad \text{for all } k > i$$

$$d^{-i}(i) = \max\{a(k) + S_{-i\theta}, d(k-1) + S_{-i\theta}\}$$

$$d^{-i}(k) = \max\{a(k) + S_k, d^{-i}(k-1) + S_k\} \quad \text{for all } k > i. \quad (37)$$

TABLE I

q	r(l=1)	Nr/Nc*	dE[T]/dq	DPA Mean (Var)	SPA Mean (Var)
0.2	0.62	100/500	1.4157	1.39619(0.100597)	1.37187(0.109193)
0.2	0.62	200/500	1.4157	1.40181(0.099802)	1.39485(0.129505)
0.3	0.65	100/500	0.798639	0.7921 (0.056603)	0.7916 (0.070113)
0.3	0.65	200/500	0.798639	0.79509(0.053081)	0.79534 (0.072530)
0.5	0.65	30/1000	-0.798639	0.79770(0.022511)	-0.81832(0.026849)
0.5	0.65	200/500	-0.798639	-0.7951 (0.053081)	-0.7952(0.072534)
0.6	0.62	200/600	-1.4175	-1.39635(0.073884)	-1.39739 (0.070673)
0.6	0.62	40/1000	-1.4175	-1.38535(0.0369313)	-1.39563(0.036392)
0.7	0.57	30/1000	-1.78983	-1.71851(0.034805)	-1.73729(0.033423)
0.7	0.57	40/1000	-1.78983	-1.7359(0.0332479)	-1.74807(0.031902)

Table I shows that overall the DPA estimate formula provides a slightly smaller variance than the SPA. On the other hand, even though the DPA estimate has a smaller variance than the SPA, the DPA estimate needs slightly more computational effort than the SPA. This means that there is a trade-off between accuracy and the computation effort.

We call the reader's attention to the fact that for this problem, to be able to calculate the derivative estimate, not only is the original sample path used, but also the new service times S_{+i0} and S_{-i0} are needed to obtain L_{+i0} and L_{-i0} (or $d^{+i}(k)$ and $d^{-i}(k)$). This is different from deleting a customer for the $G/G/1$ admission control in which the derivative estimate can be obtained by only manipulating the original sample path. Because L_{+i0} (or L_{-i0}) is different from the original sample path only at the i th customer service time, parallel simulation may be considered to implement the DPA algorithm. This topic is beyond the scope of our paper and deserves further research.

V. CONCLUDING REMARKS

In this paper we have presented the DPA. By using the step function to represent the discontinuous sample performance function, we provide a unified framework for constructing the derivative estimation of a threshold-type of problem. As we can see, there are two types of dependencies of performance function on the parameter θ . One is through θ itself, and another is through $\sigma(\xi)$. The dependence of the performance function on θ itself contributes to the IPA estimate which is the first term of the DPA estimate. The dependence on parameter θ through $\sigma(\xi)$ contributes to the second term of the DPA estimate. It is clear that if the second term of the DPA is equal to zero, then the IPA algorithm is valid, even though the sample performance function is discontinuous with respect to θ . We have also explored the relationships between the DPA and other PA techniques. It turns out that DPA is the conditional expectation of some other PA techniques such as SPA or LR. Because of this fact, we have illustrated that many new derivative estimate formulas can be derived. Finally, variance comparison between the DPA and other PA techniques is presented.

We believe that the usefulness and power of the DPA method can be extended to more general discrete-event systems. An initial result [17] has shown a positive sign. Further research will focus on the extension of the DPA to more general DEDS and on the implementing DPA algorithms with parallel simulations.

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REFERENCES

- [1] P. Brémaud and F. Vázquez-Abad, "On the pathwise computation of derivatives with respect to the rate of point process: The phantom RPA method," *Queueing Syst.*, vol. 10, pp. 249–270, 1992.
- [2] P. Brémaud and W.-B. Gong, "Derivatives of likelihood ratios and smoothed perturbation analysis for the routing problem," *ACM Trans. Modeling Computer Simulation*, vol. 3, pp. 134–161, 1993.
- [3] C. G. Cassandras, *Discrete Event Systems: Modeling and Performance Analysis*. Homewood, IL: Irwin, 1993.
- [4] L. Dai and Y. C. Ho, "Structural infinitesimal perturbation analysis (SIPA) for derivative estimation in discrete event systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1154–1166, 1995.
- [5] M. C. Fu and J. Q. Hu, "Extensions and generalizations of smoothed perturbation analysis in a generalized semi-Markov framework," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1483–1500, 1992.
- [6] A. A. Gaivoronski, L. Shi, and R. S. Sreenivas, "Augmented infinitesimal perturbation analysis: An alternate explanation," *Discrete Event Dynamic Syst.: Theory Appl.*, vol. 2, pp. 121–138, 1992.
- [7] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. 1. New York: Academic, 1964.
- [8] P. Glasserman and D. Yao, "Algebraic structure of some discrete event systems and their applications," *Discrete Event Dynamic Syst.: Theory Appl.*, vol. 1, pp. 7–36, 1991.
- [9] P. Glasserman, *Gradient Estimation Techniques for Discrete Event Systems*. Norwell, MA: Kluwer, 1991.
- [10] P. Glasserman and W.-B. Gong, "Smoothed perturbation analysis for a class of discrete event systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1218–1230, 1990.
- [11] P. W. Glynn, "Likelihood ratio gradient estimation: An overview," in *Proc. Winter Simulation Conf.*, 1987, pp. 366–375.
- [12] W.-B. Gong and Y. C. Ho, "Smoothed perturbation analysis of discrete-event dynamic systems," *IEEE Trans. Automat. Contr.*, vol. 32, pp. 858–867, 1987.
- [13] Y. C. Ho and X. R. Cao, *Discrete Event Dynamic Systems and Perturbation Analysis*. Norwell, MA: Kluwer, 1991.
- [14] P. L'Ecuyer, "A unified view of the IPA, SF, and LR gradient estimation techniques," *Management Sci.*, vol. 36, pp. 1364–1383, 1990.
- [15] M. I. Reiman and A. Weiss, "Sensitivity analysis for simulation via likelihood ratios," *Operations Res.*, vol. 37, pp. 830–844, 1989.
- [16] R. Y. Rubinstein, "Sensitivity analysis and performance extrapolation for computer simulation models," *Operations Res.*, vol. 37, pp. 72–81, 1989.
- [17] L. Shi, "On the consistency of high order derivative estimators for the M/G/1 queue," in *Proc. Thirty-second Ann. Allerton Conf. Communication, Contr., and Computing*, 1994, pp. 683–690.
- [18] R. Suri and X. Cao, "The phantom customer and the marked customer methods for optimization of closed queueing networks with blocking and general service times," *ACM Performance Evaluation Rev.*, vol. 12, pp. 243–256, 1986.